

# Classical, Covariant Finite-size Model of the Radiating Electron

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The finite extension of the classical electron is defined in a new, covariant manner. This new definition enables one to calculate exactly the bound and emitted four-momentum and to find an equation of motion different from the Lorentz-Dirac equation and from other equations proposed in the literature. Neither mass renormalization nor use of advanced quantities nor asymptotic conditions are necessary. Runaway solutions and pre-acceleration do not occur in the framework of the model presented here.

## I. Introduction

Although interest of present-day physics is concentrated mainly in the field of quantum mechanics, the classical theory of the radiation-damped electron has lost nothing of its importance. This is due to the fact, that the radiation of free accelerated electrons is the key to understand some phenomena in modern astrophysics (e.g. pulsars) and that new and powerful accelerators seem now to be available in order to test various equations of motion for charged particles including radiation-damping<sup>1–3</sup>.

Now, the theory, which is used in the literature to describe the trajectory of a radiating electron, is the classical Lorentz-Dirac theory\* based on Dirac's equation<sup>4, \*\*</sup>

$$m_0 c^2 \dot{u}^\mu = Z F_{\text{ext}}^{\mu\nu} u_\nu + \frac{2}{3} Z^2 [\ddot{u}^\mu + (\dot{u} \dot{u}) u^\mu]. \quad (1)$$

But it is known for a long time that there are some weak points in this theory – as well in the derivation of Eq. (1) as in its application to practical problems – and as a consequence a considerable discussion arose in the literature. Let us remind only of mass renormalization, runaway solutions, pre-acceleration, and use of advanced quantities. Therefore, there has been in recent time no lack of attempts either to put Eq. (1) on a more reliable basis (Teitelboim<sup>6</sup>, Synge<sup>7</sup>, Hogan<sup>8</sup>, Barut<sup>9</sup>) or to propose a new equation in stead of (1) (Eliezer<sup>10</sup>, Caldirola<sup>11</sup>, Bonnor<sup>12</sup>, Mo + Papas<sup>13</sup>).

The characteristic feature, which is common to all the proposed equations in<sup>10–13</sup>, is that these new equations have been set up on a purely postu-

lational basis. The authors seem to feel that on the ground of Maxwell's theory of classical electromagnetism one cannot go farther than Dirac has done in his famous work<sup>4</sup>. Therefore the only possibility to come to a new, improved equation of motion is left over to postulation.

Well, we do not agree with this point of view, but we shall show, that it is possible to build up a theory of an extended electron from Maxwell's theory of electromagnetism, which has to be exceeded only by a few intuitive and plausible ideas. Dirac himself has felt that the solution to the problem might arise from taking into account the extension of the electron in a correct manner. He writes<sup>4</sup>: "The finite size of the electron now reappears in a new sense, the interior of the electron being a region of failure, not of the field equations of electromagnetic theory, but of some of the elementary properties of space-time."

It is the aim of this paper to work out such a theory of an extended electron being compatible with the fundamental principles of special relativity.

After having defined the electron's extension in a new way, it shall be possible to calculate exactly the bound and emitted four-momentum of the extended particle. Some of the results available in the literature and valid for the point-like particle are contained in the more general theory of finite extension. An equation of motion for the extended particle can be found which, however, does not lead in its lowest-order approximations to the Lorentz-Dirac Eq. (1) nor to the postulated equations<sup>10–13</sup>.

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\* An introduction to this theory with all relevant references up to 1965 can be found in Rohrlich's book<sup>5</sup>, as well as many conventions and notations, used in this paper.

\*\* The line-element is written here as  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (dx^0)^2 - (d\mathbf{x})^2$ . Four-velocity  $u^\lambda = dz^\lambda/ds$ , acceleration  $du^\lambda/ds \equiv \ddot{u}^\lambda$  etc. Sometimes a four-vector  $\{a^\lambda\}$  is decomposed as  $\{a^\lambda\} = \{a^0; \mathbf{a}\}$ . Scalar product  $a^\lambda b_\lambda = (a \cdot b) = a^0 b^0 - (\mathbf{a} \cdot \mathbf{b})$ .



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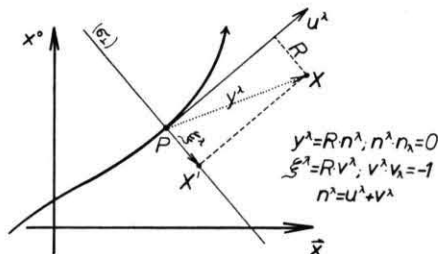
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## II. Some geometrical and physical preliminaries

Recently, Synge<sup>7</sup> has introduced a new coordinate system in space-time working excellently in describing relativistic electron kinematics. We shall need, however, only a few elementary notations of this new coordinate system, which we shall sketch briefly:



The world-line  $x^\mu = z^\mu_a$  of the (point-like) particle being given, an invariant distance  $R$  from this world-line can be assigned to every point  $X\{x^0, x^1, x^2, x^3\}$  of space-time by (see Fig. 1)

$$R = u^\mu \cdot (x_\mu - z_\mu)$$

with

$$(x - z)^\lambda \cdot (x - z)_\lambda = 0.$$

The point  $P$  is the intersection of the world-line and the backward light cone with vertex in  $X$ .

The remaining two coordinates of  $X$  are obtained by projecting  $X$  on to the orthogonal hyperplane  $\sigma_{\perp(P)}$  to the world-line in  $P$ . The projected point  $X'$  has the coordinates  $(\xi^1, \xi^2, \xi^3)$  in an orthonormal triad of  $\sigma_{\perp(P)}$ .

Clearly, we can introduce spherical coordinates in  $\sigma_{\perp}$ :

$$\xi^1 = R \sin \Theta \cos \Phi, \xi^2 = R \sin \Theta \sin \Phi, \xi^3 = R \cos \Theta. \quad (2)$$

The retarded distance  $R$  plays an important part for the following considerations.

As is well known from the standard theory of electromagnetism, the point-like electron is described by the Liénard-Wiechert potentials  $A^\lambda$  (note, that we use exclusively the retarded potentials):

$$A^\lambda(X) = Z u_{(p)}^\lambda / R.$$

With these potentials we find the field strengths

$$\begin{aligned} F^{\mu\nu} &\equiv A^{\mu/\nu} - A^{\nu/\mu} \equiv 2 A^{[\mu/\nu]} \\ &= \frac{2Z}{R^2} [1 - R(v \dot{u})] \cdot u^{[\mu} v^{\nu]} + \frac{2Z}{R} \dot{u}^{[\mu} n^{\nu]} \end{aligned} \quad (3)$$

(note the interchanged role of  $u^\lambda$  and  $v^\lambda$  with respect to the notation of Rohrlich<sup>5</sup>).

Finally we obtain the energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{4\pi} [-F_{\mu}{}^\lambda F_{\nu\lambda} + \frac{1}{2} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}] \quad (4)$$

as

$$T_{\mu\nu} = {}_I T_{\mu\nu} + {}_{II} T_{\mu\nu} \quad (5)$$

with

$$\begin{aligned} -4\pi \cdot {}_I T_{\mu\nu} &= \frac{Z^2}{R^4} \{n_\mu n_\nu - 2u_{(\mu} n_{\nu)} + \frac{1}{2} g_{\mu\nu}\} \\ &+ \frac{2Z^2}{R^3} \{u_{(\mu} n_{\nu)} - n_\mu n_\nu\} \cdot (v \dot{u}) \end{aligned} \quad (6a)$$

and

$$-4\pi {}_{II} T_{\mu\nu} = \frac{Z^2}{R^2} \{ (v \dot{u})^2 + (\dot{u} \dot{u}) \} n_\mu n_\nu. \quad (6b)$$

Teitelboim<sup>6</sup> has shown that

$$\partial_\mu {}_I T^{\mu\nu} \equiv 0; \quad \partial_\mu {}_{II} T^{\mu\nu} \equiv 0 \quad (7)$$

is valid off the world-line and thus Eq. (5) provides a natural decomposition of the whole energy-momentum (4) into two parts. The first one ( ${}_I T_{\mu\nu}$ ) can be identified with the bound energy-momentum density and the second part with the emitted density.

The four-momentum  $P_{\text{part}}^\lambda$  of the electron is usually defined as<sup>5</sup>

$$P_{\text{part}}^\lambda = \frac{1}{c} \int_{\sigma_{\perp}} T_{\nu}{}^\lambda d^3 \sigma_{\perp}^\nu \quad (8)$$

with  $T_{\nu}{}^\lambda$  taken from the Eqs. (6), and

$$d^3 \sigma_{\perp}^\nu = u^\nu d^3 \sigma_{\perp}.$$

The invariant element  $d^3 \sigma_{\perp}$  reduces to the volume element of ordinary three-space

$$d^3 \sigma_{\perp} \rightarrow dx^1 dx^2 dx^3$$

in the rest system of the electron. The integration in (8) runs over the whole plane  $\sigma_{\perp}$  and therefore sums up the whole history of the particle till the world-point, in which the four-momentum is being calculated. One could suppose now, that the four-momentum  $P_{\text{part}}^\lambda$ , thus defined, is a functional of the whole backward world-line. Later on, we shall show that this is true only for the emitted part  $P_{\text{rad}}^\lambda$  but not for the bound part  $P_{\text{b}}^\lambda$ , the sum of which is given by  $P_{\text{part}}^\lambda$ .

Besides the plane  $\sigma_{\perp}$  there is another hypersurface, which shall be used to perform the integration in (8). This is the future light cone  $l_{(P)}$

with vertex in P:

$$Q^i = \frac{1}{c} \int_{l(P)} T_r^i d^3 l^r. \quad (8a)$$

The hypersurface element  $d^3 l_r$  on the light cone is given by the projection formula of Synge<sup>7</sup>

$$d^3 l^r = n^r R^2 dR d\Omega \quad (9)$$

where  $d\Omega$  is the solid-angle element in the plane  $\sigma_\perp$  [note (2)]:

$$d\Omega = \sin \Theta d\Theta d\Phi; \quad (10)$$

$Q^i$  is the electromagnetic flux through the light cone  $l_{(P)}$ . Of course, we expect that  $Q^i$  is a local quantity (contrary to  $P_{\text{part.}}^\lambda$ ), i.e.  $Q^i$  depends only on kinematical quantities of the world-line in P.

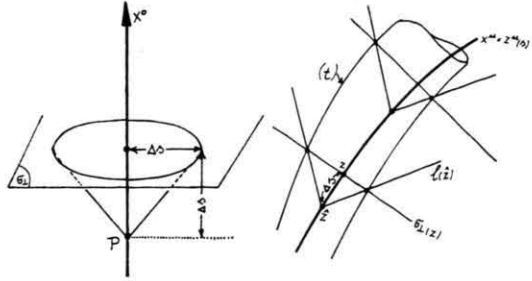
### III. How to define the "extension" of the electron?

Dirac, too, had to conceive some idea, how the extension of the electron enters his theory, because he treats the point-like particle as limit of an extended particle with vanishing size. He defines the electron's surface to be a sphere with constant radius in the rest-system of the particle, i.e. he cuts out a hole (of constant radius) of every hyperplane  $\sigma_\perp$  along the world-line. This seems to be incompatible with the finite velocity of propagation of any information, for the information going out of the source as the centre of the sphere and telling the surface to be a sphere around this centre requires some finite time, whereas in Dirac's theory the information is transmitted instantaneously.

If we now look for a new definition of the extension, we should start with the two requirements:

- The "true" extension is seen by an observer in the rest-system of the particle (otherwise the Lorentz contraction deforms the shape of the particle); i.e. the surface of the electron is constituted by world-points which ly in the instantaneous rest plane  $\sigma_\perp$  of the source.
- In order to be consistent with special relativity, the extension should be defined by means of light signals.

Now, we know that when the source has ever been at rest and emits in a certain world-point P (see Fig. 2) an electromagnetic signal, this signal



has reached after the time  $\Delta t = \Delta s/c$  the shape of a sphere in the plane  $\sigma_\perp(s_{(P)} + \Delta s)$  with radius  $\Delta s$ . By generalizing these findings to an arbitrarily curved world-line of the source we define the surface of the electron at proper time  $s = c \cdot \tau$  to be the intersection of the future light-cone  $l_{(z(s-\Delta s))}$  with vertex in  $x^\mu = z^\mu_{(s-\Delta s)} =: \hat{z}^\mu$  and the plane  $\sigma_\perp(z_{(s)})$  (see Fig. 3).

If we shift the hypersurfaces  $l(\hat{z})$  and  $\sigma_\perp(z)$  along the world-line, holding the arc-length distance  $\Delta s$  fixed, we construct a tube  $(t)$ , the interior of which must be regarded as inaccessible to any classical theory.

In the geometrical sense, we may expect in the electron's interior thus defined a breakdown of Euclidean Geometry (if you want, think of some sort of a black hole) and in the quantum-mechanical sense the interior of the tube is governed by quantum-electrodynamics but no longer by the classical Liénard-Wiechert potentials<sup>†</sup>.

### IV. Elementary, differential-geometrical remarks on the extension of the electron

In Sect. III we have defined the "surface" of the electron at some instant as the intersection of the future light cone  $l_{(\hat{z})}$  and the hyperplane  $\sigma_\perp(z)$ . Therefore the equation determining the tube  $(t)$  can be obtained by eliminating the arc-length  $s$

<sup>†</sup> A simple consideration of Heisenberg's uncertainty relation together with the requirement, that the mass uncertainty should not surpass the mass difference between the electron and the next heavier particle ( $\mu$ -meson), delivers a lower bound to the measure time  $\Delta t$  respectively to the minimal distance  $\Delta s = c \cdot \Delta t$ , at which a classical description of the electron's immediate surroundings is allowed. One finds  $\Delta s \approx r_0$ , where  $r_0$  is the classical electron radius.

from the two equations

$$l_{(\hat{z})} : (\xi - \hat{z})^2 = 0, \quad (11 \text{ a})$$

$$\sigma_{\perp(z)} : u \cdot (\xi - z) = 0 \quad (11 \text{ b})$$

$[\xi^i$  is some point on the tube  $(t)$ ].

In vector-notation the tube  $(t)$  is given by (note Fig. 4) <sup>††</sup>

$$\xi = \hat{z} + R_{\min} \cdot \hat{n}. \quad (12)$$

For the sake of uniqueness we shall use the angles  $\hat{\theta}, \hat{\phi}, d\hat{\Omega}$ .

Let us now calculate the angle-dependent distance  $R_{\min}(\hat{\theta}, \hat{\phi})$ . From (12) we obtain first

$$R_{\min} = (\xi - \hat{z}) \cdot \hat{u}$$

and from (11 b) and (12) we find

$$u \cdot (\hat{z} - z + R_{\min} \cdot \hat{n}) = 0.$$

Therefore, abbreviating  $z - \hat{z} = : \mathcal{Z}$ , one gets

$$R_{\min} = (\mathcal{Z} \cdot u) / (\hat{n} \cdot u). \quad (13)$$

Now we ask, what is the shape of the electron in the rest system of the source. To answer this question, we project the vector  $\xi$  on to the "now-plane"  $\sigma_{\perp(z)}$  of a comoving observer [note (12) and (13)]:

$$\xi_{\sigma\perp} \equiv \xi - (\xi \cdot u) u = \hat{z} - (z \cdot \hat{u}) u + (\mathcal{Z} u) \cdot \left[ \frac{\hat{n}}{(\hat{n} \cdot u)} - u \right]$$

The space-like unit vector  $m$

$$m := \hat{n} / (\hat{n} \cdot u) - u; \quad m^2 = -1 \quad (14)$$

lies in the plane  $\sigma_{\perp(z)}$  and is the normal of the intersection  $l_{(\hat{z})} \wedge \sigma_{\perp(z)}$ . Therefore

$$\{\xi_{\sigma\perp} - [\hat{z} - (z \cdot \hat{u}) u]\}^2 = (\mathcal{Z} u)^2 \cdot m^2 = -(\mathcal{Z} u)^2 = : -\varrho^2. \quad (15)$$

Since the right-hand side of (15) is angular-independent, we see that the surface of the electron in the instantaneous rest-system of the source is a sphere of radius  $\varrho = (\mathcal{Z} u)$ . The four-dimensional radius-vector  $\vec{OM}$  of the center M of this sphere is

$$\vec{OM} = \hat{z} + R_{\min} \cdot \hat{n} - \varrho m = \hat{z} + \varrho \cdot u. \quad (16)$$

From here it is seen that the center M of the sphere is not identical with the source, in general, but runs on an extra world-line, which is slightly displaced from that of the source.

<sup>††</sup> All quantities with a bar, like  $\hat{z}, \hat{u}, \hat{n}$  etc., are referred to the advanced point  $z(s - \Delta s)$  on the world-line.

The radius  $\varrho$  changes along the world-line. If we expand  $\hat{z}$  in

$$\varrho = (\mathcal{Z} u) = (z - \hat{z}) \cdot u$$

about the point  $z$ , we find

$$\varrho \cong \Delta s \cdot [1 - \frac{1}{6} (\Delta s)^2 \cdot (\dot{u} \cdot \dot{u})]$$

and therefore  $\varrho$  takes on its minimal value ( $= \Delta s$ ) in the case of uniform motion.

In order to derive an equation of motion for the extended particle, it shall be necessary to know the hypersurface element  $d^3 f^\mu$  of the tube  $(t)$ , playing an analogous part as  $d^3 \sigma_{\perp}^\nu$  in (8) or  $d^3 l^\nu$  in (8 a). The derivation of  $d^3 f^\mu$  requires a more subtle differential-geometrical consideration and is therefore carried through in Appendix A. The result is

$$d^3 f^\nu = ds \cdot \frac{R_{\min}^2}{(\hat{n} \cdot u)} d\hat{\Omega} \{ -u^\nu + [1 - \dot{u} \cdot (\xi - z)] \hat{n}^\nu \}. \quad (A 11)$$

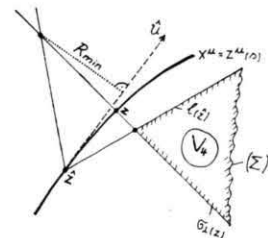
## V. Exact, direct calculation of the bound and emitted four-momentum of the extended particle

The total electromagnetic four-momentum  $P_{\text{part.}}^\mu$  of the electron is the sum of the bound part  $P_b^\mu$  and the emitted part  $P_{\text{rad}}^\mu$

$$P_{\text{part.}}^\mu = P_b^\mu + P_{\text{rad}}^\mu. \quad (17)$$

$P_{\text{part.}}^\mu$  is calculated from the relation (8) with the energy-momentum tensor given by (5–6 b). The integration in (8) runs over the hyperplane  $\sigma_{\perp(z)}$ , if  $P_{\text{part.}}^\mu$  is to be determined in the world-point  $z$ ; but, due to our definition of the extension of the electron, that part of  $\sigma_{\perp(z)}$ , which is cut out by the light cone  $l_{(\hat{z})}$ , must be omitted in the integration.

We obtain the desired hypersurface integral (8) very conveniently and simultaneously with the splitting (17), if we apply Gauß' integral-theorem to the four-dimensional domain  $V_4$ , enclosed by the three hypersurfaces  $\sigma_{\perp(z)}$ ,  $l_{(\hat{z})}$  and  $\Sigma$  (see Figure 4).  $\Sigma$





is some arbitrary hypersurface, which we can shift to infinity.

The conservation law (7) reads then in integral form

$$\int_{l(\hat{z})} T_v^\mu d^3 l_{(\hat{z})}^\nu + \int_{\Sigma} T_v^\mu d^3 \Sigma^\nu = \int_{\sigma_{\perp(z)}} T_v^\mu d^3 \sigma_{\perp(z)}^\nu. \quad (18)$$

From here we see, that the desired integral on the right-hand side of (18) is composed of the flux (8a) through the light-cone  $l(\hat{z})$  and of the flux through the infinitely distant surface  $\Sigma$ . It must be clear, therefore, that this last part is necessarily the total emitted four-momentum  $P_{\text{rad}(z)}^\mu$ , to which all world-points before  $\hat{z}$  have contributed. In order that (18) be meaningful, we require the initial condition

$$\lim_{s \rightarrow -\infty} u^\lambda = \text{constant}. \quad (19)$$

If we choose for  $\Sigma$  the hypersurface  $R = \text{const}$  with  $\text{const} \rightarrow \infty$ , bearing in mind the sondary condition  $s \leq \hat{s}$ , then the result of integration can be taken over from Sygne<sup>7</sup>:

$$\frac{1}{c} \int_{R=\text{const.} \rightarrow \infty} T_v^\mu d^3 \Sigma^\nu = - \frac{1}{c} \frac{2}{3} Z^2 \int_{-\infty}^{\hat{s}} (\dot{u} \dot{u}) u^\mu ds \equiv P_{\text{rad}(z)}^\mu. \quad (20)$$

The remaining integral on the left-hand side of Eq. (18), running over the light cone  $l(\hat{z})$ , has to be identified – due to (17) – with the bound four-momentum of the electron

$$P_{b(z)}^\mu \equiv \frac{1}{c} \int_{l(\hat{z})} T_v^\mu d^3 l_{(\hat{z})}^\nu. \quad (21)$$

Of course,  $P_{b(z)}^\mu$  is a local quantity – contrary to  $P_{\text{rad}}^\mu$  – because the energy-momentum tensor on the surface  $l(\hat{z})$  depends only on geometrical quantities of the world-line in  $\hat{z}$ . Since the integration in (21) brings in the world-point  $z$ , we expect, that  $P_{b(z)}^\mu$  is a function of  $z$  and  $\hat{z}$  only, but not a function of the total prehistory of the particle.

The explicit integration in (21) is done very easily, if we note that with (6b) and (9)

$${}_{\text{II}} \hat{T}_v^\mu \hat{n}^r = 0 \quad (22)$$

and therefore

$$P_{b(z)}^\mu = \frac{1}{c} \int_{l(\hat{z})} {}_{\text{I}} \hat{T}_v^\mu d^3 l_{(\hat{z})}^\nu. \quad (23)$$

Inserting here (6a), we find [note (13)]:

$$P_{b(z)}^\mu = \frac{1}{c} \frac{Z^2}{8\pi} \int d\hat{\Omega} \hat{n}^\mu \int_{R_{\text{min}}}^{\infty} \frac{dR}{R^2} \\ = \frac{1}{c} \frac{Z^2}{2\varrho} \left\{ \frac{4}{3} \hat{u}^\mu (\hat{u} u) - \frac{1}{3} u^\mu \right\}. \quad (24)$$

For a particle in uniform motion the expression (24) reduces to the well-known form

$$P_{b'}^\mu \rightarrow \frac{1}{c} \frac{Z^2}{2 \cdot \Delta s} u^\mu \quad (25)$$

which is the justification for having put ( $\Delta s$ ) equal to  $r_0$  in Sect. III, for  $Z^2/2r_0 = m_0 c^2$  is the old result from standard electrodynamics.

To Eq. (23) we may add the analogous statement for the radiative part

$$P_{\text{rad}(z)}^\mu = \frac{1}{c} \int_{\Sigma} {}_{\text{II}} T_v^\mu d^3 \Sigma^\nu \quad (26)$$

because the short-range density  ${}_{\text{I}} T_v^\mu$  does not contribute anything on ( $\Sigma$ ).

(23) and (26) confirm and generalize the result of Teitelboim<sup>6</sup> which says, that  ${}_{\text{I}} T^{\mu\nu}$  builds up  $P_{b'}^\mu$  and  ${}_{\text{II}} T^{\mu\nu}$  builds up  $P_{\text{rad}}^\mu$ , each separately, and that  $P_{b'}^\mu$  is a local quantity. Because of the conservation laws (7) and (22) we can equally well formulate

$$P_b^\mu = \frac{1}{c} \int_{\sigma_{\perp}} {}_{\text{I}} T_v^\mu d^3 \sigma_{\perp}^\nu, \quad (27a)$$

$$P_{\text{rad}}^\mu = \frac{1}{c} \int_{\sigma_{\perp}} {}_{\text{II}} T_v^\mu d^3 \sigma_{\perp}^\nu \quad (27b)$$

which shows more intimately the connection with Teitelboim's work. Van Weert<sup>14</sup> has proofed the local feature of (27a) for the point-like case in full mathematical rigour, showing that  ${}_{\text{I}} T^{\mu\nu}$  is the divergence of a tensor of higher rank and applying Stokes' theorem in space-time. We see that our model makes such a proof completely superfluous, but incorporates the local feature of  $P_{b'}^\mu$  a priori and not only in the limit of vanishing size but in the case of an arbitrary extension. Let us close this section by the remark that Teitelboim<sup>6</sup> was the first to realize that the four-momentum of the electron in non-uniform motion differs from its value in uniform motion. He finds for the point-like electron

$$P_{b'}^\mu = \frac{1}{c} \frac{Z^2}{2r_0} u^\mu - \frac{2}{3} \frac{Z^2}{c} \dot{u}^\mu \quad (28)$$

i. e. the well-known expression (25) is generalized by the Schott-term. Clearly, we can reproduce this result by inserting in (24) the expansion

$$\hat{u}^\mu = u^\mu - \Delta s \cdot \dot{u}^\mu \dots$$

and retaining only terms of maximal order  $(\Delta s)^0$ .

## VI. The origin of the Schott-term

If the attempts, to give some physical meaning to the Schott-term, are pursued in the literature, one gets the impression that the origin of the Schott-term is obscure up to the present day. For the first time, the Schott-term emerges in Abraham's work<sup>15</sup> as part of the generalization to the non-relativistic radiation-reaction term

$$\frac{2}{3} \frac{Z^2}{c^3} \frac{d^2 \mathbf{v}}{dt^2} \rightarrow \frac{2}{3} Z^2 [\ddot{u}^\lambda + (\dot{u}^\lambda \dot{u}^\lambda) u^\lambda].$$

Even in our days, this way of argumentation is followed by Landau-Lifschitz in their book<sup>16</sup>. It is clear that in this procedure the Schott-term must be regarded as a radiation-reaction term. But, as Rohrlich<sup>5</sup> states, this interpretation has to be rejected, because the radiation-recoil is

$$-\frac{dP_{\text{rad}}^\mu}{ds} = \frac{2}{3} \frac{Z^2}{c} (\dot{u}^\lambda \dot{u}^\lambda) u^\mu$$

as is well-known from standard electromagnetic theory. Schott<sup>17</sup> has given to the time-component

$$\frac{2}{3} Z^2 \frac{\gamma^4}{c^3} \left( \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \right); \quad \left\{ \gamma = \left[ 1 - \left( \frac{\mathbf{v}}{c} \right)^2 \right]^{-1/2} \right\} \quad (29)$$

of the term, bearing his name, the meaning of an "acceleration energy" without specifying the origin of this acceleration energy. Dirac<sup>4</sup> himself subscribes to the view of Schott by stating that "changes in the acceleration energy correspond to a reversible form of emission or absorption of field-energy, which never gets very far from the electron." Now, the Schott-term has been derived in Dirac's work by purely classical means and one should ask therefore, by which feature of the Liénard-Wiechert fields such reabsorption processes are managed?

Syngé<sup>7</sup> gives a derivation of the Lorentz-Dirac equation in which the Schott-term appears as a consequence of some short-range forces of non-electromagnetic nature, but one feels that the preference should be given to that interpretation of the Schott-term, which flows from the ordinary Maxwell-

theory or at least exceeds this theory by a minimal number of additional assumptions.

Steinwedel<sup>18</sup> has supposed – and this will be confirmed partially in the framework of our model – that the Schott-energy is due to the fact, that the field in the immediate neighbourhood of the singularity has been emitted short time before the moment when it contributes to the energy-momentum of the particle. Therefore, with (29) and (28), the energy of the electron is diminished by the amount (29), if  $\mathbf{v}$  parallel to  $d\mathbf{v}/dt$ , and increased, if  $\mathbf{v}$  antiparallel to  $d\mathbf{v}/dt$ .

Whereas Steinwedel's argumentation was non-relativistic, Hogan<sup>8</sup> has tried to give a covariant derivation of the Lorentz-Dirac equation, in which the Schott-term appeared after a series-expansion of an advanced term of inertia:

$$\frac{dP_{\text{b}}^\mu}{ds} = \lim_{\varepsilon \rightarrow 0} \frac{Z^2}{2c\varepsilon} \dot{u}_{(s-\frac{1}{2}\varepsilon)}^\mu \rightarrow \frac{1}{c} \frac{Z^2}{2\varepsilon} \dot{u}_{(3)}^\mu - \frac{2}{3} \frac{Z^2}{c} \dot{u}_{(s)}^\mu. \quad (30)$$

As attractive as this idea might be, Hogan's procedure leading to the result (30) is rather inconsistent: In calculating the flux through the infinitely distant hypersurface  $R = \text{const}$ , he anticipates the limit  $\text{const} \rightarrow \infty$ , whereby this surface becomes a null-surface with normal  $\{n^\nu\}$ . Therefore Eq. (22) should apply and Hogan finds the result that the flux through the surface mentioned above vanishes. It is clear that this conclusion is not correct, because the flux through the infinitely distant surface has to be calculated at finite value of  $R$  and then  $R$  may be shifted to infinity, thus obtaining the value  $P_{\text{rad}}^\mu$  of Section V. In equating the flux through the infinitely distant surface to zero, one would have completely disregarded the radiation phenomenon (this error is compensated in Hogan's work by the coupling of two independent limiting-procedures).

Let us study now the emergence of the Schott-term in the context of our model:

To this end, we first note that if we do not take care of the shape of the electron and if we make in (24) the identification  $R_{\text{min}} \equiv r_0$ , then we get indeed

$$\begin{aligned} P_{\text{b(z)}}^\mu &= \frac{1}{c} \frac{Z^2}{2r_0} \int \frac{d\hat{\Omega}}{4\pi} \hat{n}^\mu = \frac{1}{c} \frac{Z^2}{2r_0} \hat{u}^\mu \\ &= \frac{1}{c} \frac{Z^2}{2r_0} u_{(s-r_0)}^\mu \simeq \frac{1}{c} \frac{Z^2}{2r_0} u_{(s)}^\mu - \frac{1}{2} \frac{Z^2}{c} \dot{u}_{(s)}^\mu \dots \end{aligned}$$

From here we see that the part

$$\frac{1}{2} \frac{Z^2}{c} \dot{u}^\mu \quad (31)$$

of the Schott-term is indeed due to an advanced term of inertia and therefore Steinwedel's result is justified partially.

But the remainder

$$\frac{1}{6} \frac{Z^2}{c} \dot{u}^\mu \quad (32)$$

appears now to be a purely geometrical surface-effect, which is due to the angular dependence of  $R_{\min}$ . This surface-effect consists in the fact, that in Eq. (24) the angular integration has to be performed in the now-plane of the source at that instant when the light signal, defining the surface of the electron at time  $\Delta\tau = \Delta s/c$  later, is emitted from the source; whereas the true surface, at the instant of integration, is defined by a light signal which was emitted at time  $\Delta\tau$  earlier. Notably, the latter surface is a sphere in  $\sigma_{\perp}(\hat{s})$ , whereas the orthogonal projection of the first sphere on to the plane  $\sigma_{\perp}(\hat{s})$  is the ellipsoid

$$R_{\min} = \frac{\varrho}{(\hat{n}u)} = \frac{(\varrho/\tilde{u}^0)}{1 - (|\mathbf{u}|/|\tilde{\mathbf{u}}|) \cos \Theta}$$

( $\tilde{u}^k$  is  $u^k$ , measured in the rest-system at the world-point  $\hat{z}$ ).

The anisotropy of this ellipsoid causes, in first order, the remainder (32).

It is seen, that our model is able to provide a satisfactory explanation of the Schott-term, this term being the first-order correction to the ordinary four-momentum (25). This correction originates partially (31) from an advanced term of inertia and partially (32) from the apparent shape of the electron, being different from the true shape in the case of non-uniform motion.

## VII. The electromagnetic flux through the tube-surface ( $t$ )

In order to prepare an equation of motion to the extended electron, it is advisable to study the flux ( $T^\mu$ ) through the tube-surface ( $t$ ).

One can do this in two ways:

- indirect calculation by means of Gauß' integral theorem,
- direct calculation by application of the hyper-surface element  $d^3j^\mu$  from (A 11).

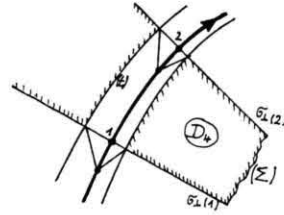
### a) Indirect calculation

From the vanishing of the tensor divergence (7) we conclude – similar to the considerations of Section V – for the domain  $D_4$  (see Fig. 5)

$$P_{\text{part.}(1)}^\mu - P_{\text{part.}(2)}^\mu + T_{(1,2)}^\mu = 0 \quad (33)$$

with

$$T_{(1,2)}^\mu = -\frac{1}{c} \int_1^2 \int_{(t)} T_\nu^\mu d^3j_\nu.$$



For the choice of  $\Sigma$  see the statements with respect to Figure 4. Teitelboim<sup>6</sup> has shown that the flux through  $\Sigma$  vanishes, if one assumes the asymptotic condition (19). First we retain this condition in order that  $P_{\text{part.}}^\mu$  be meaningful, but later on we can renounce on it.

Because of the splitting (5–7) we can exacerbate the relation (33) to give

$$P_{b(1)}^\mu - P_{b(2)}^\mu + T_{b(1,2)}^\mu = 0, \quad (33 a)$$

$$P_{\text{rad}(1)}^\mu - P_{\text{rad}(2)}^\mu + T_{\text{rad}(1,2)}^\mu = 0 \quad (33 b)$$

with

$$T_{b(1,2)}^\mu = -\frac{1}{c} \int_1^2 \int_{(t)} I T_\nu^\mu d^3j_\nu,$$

$$T_{\text{rad}(1,2)}^\mu = -\frac{1}{c} \int_1^2 \int_{(t)} II T_\nu^\mu d^3j_\nu.$$

If we wish to have the differential formulation of the conservation law (33), we have to shift point 2 on the world-line towards point 1, and in doing so we see that the surface content of  $\Sigma'$  (time-like) vanishes in the limit and so does the flux through it. In the same way the infinite parts of  $P_{\text{rad}(1)}^\mu$  and  $P_{\text{rad}(2)}^\mu$  cancel and so the initial condition (19) can be abandoned. Of course, the flux-density ( $T^\mu$ ) through ( $t$ ) per world-line element  $ds$  cannot depend on this initial condition. For the flux-density

just mentioned we obtain

$$\mathcal{T}_b^\mu = \frac{dP_b^\mu}{ds} = -\frac{1}{c} \int_{\hat{\Omega}} {}_1T_v^\mu \frac{d^3f_v}{ds}, \quad (34a)$$

$$\mathcal{T}_{\text{rad}}^\mu = \frac{dP_{\text{rad}}^\mu}{ds} = -\frac{1}{c} \int {}_{11}T_v^\mu \frac{d^3f_v}{ds}, \quad (34b)$$

$$\mathcal{T}_{\text{part.}}^\mu = \mathcal{T}_b^\mu + \mathcal{T}_{\text{rad}}^\mu = \frac{dP_{\text{part.}}^\mu}{ds}. \quad (34)$$

Clearly, the values for  $P_b^\mu$ ,  $P_{\text{part.}}^\mu$ ,  $P_{\text{rad}}^\mu$  have to be inserted here from (20), (24).

### b) Direct calculation

The direct calculation of the electromagnetic flux through  $(t)$  by means of the hypersurface element  $d^3f_\lambda$  does not lead to any mathematical problem and is feasible in a completely exact manner. Since the expressions are rather lengthy, we do not reproduce the calculation here; but the results are, of course, the same as with method a).

## VIII. Equation of motion for the extended electron

In this section, we have to distinguish between the particle-field  $F_{\text{part.}}^{\mu\nu}$ , given by the Liénard-Wiechert potentials of our electron under consideration, and the external field  $F_{\text{ext.}}^{\mu\nu}$ . The total field is the sum of these two fields

$$F_{\text{tot}}^{\mu\nu} = F_{\text{part.}}^{\mu\nu} + F_{\text{ext.}}^{\mu\nu}. \quad (35)$$

Because the total energy-momentum tensor is quadratic in the field strengths, we obtain a splitting of  $T_{\text{tot}}^{\mu\nu}$  into three parts

$$T_{\text{tot}}^{\mu\nu} = T_{\text{part.}}^{\mu\nu} + T_{\text{int.}}^{\mu\nu} + T_{\text{ext.}}^{\mu\nu}. \quad (36)$$

Integration of this tensor, as indicated in (8), over some plane  $\sigma_\perp$  gives three parts of total four-momentum

$$P_{\text{tot}}^\mu = P_{\text{part.}}^\mu + P_{\text{int.}}^\mu + P_{\text{ext.}}^\mu. \quad (37)$$

In the course of the derivation of an equation of motion for the radiating electron, Haag<sup>19</sup> and Rohrlich<sup>5</sup> have excessively stressed the importance of the quantity  $P_{\text{tot}}^\mu$ , especially in the asymptotic regions  $s \rightarrow \pm\infty$ . Moreover, they did not shrink back from the use of advanced quantities, as was done by Dirac, too. This procedure seems to us to be completely unjustified. The reasons are the following:

1) On the hyperplane  $\sigma_\perp (s \rightarrow \infty)$ , Haag<sup>19</sup> defines an “outgoing” free field  $F_{\text{out}}^{\mu\nu}$  as the difference between the total, really observed (retarded) field  $F_{\text{out}}^{\mu\nu}$  and a Liénard-Wiechert-field  $F_{\text{Coul.,out}}^{\mu\nu}$  which should be due to a charge in uniform motion with  $u_{(s \rightarrow \infty)}^\lambda = \text{const}$

$$F_{\text{out}}^{\mu\nu} = F_{\text{tot}}^{\mu\nu} - F_{\text{Coul.,out}}^{\mu\nu}.$$

Now one could suppose, as Haag has done, that for the construction of  $F_{\text{Coul.,out}}^{\mu\nu}$  (as auxiliary for the construction of  $P_{\text{Coul.,out}}^\mu$ ) there were present only the advanced potentials, because these refer to the final velocity  $u_{(s \rightarrow \infty)}^\lambda$ , whereas the retarded Liénard-Wiechert potentials describe the whole earlier scattering process and therefore do not refer to the initial or final velocity. But this is not true, for in Section V we have shown that the integral

$$\frac{1}{c} \int_{\sigma_\perp} {}_1T_{(\text{ret})}^{\mu\nu} d^3\sigma_{\perp v}$$

does not depend on the pre-history of the particle, but that it is a local quantity, which can be therefore taken to define the final four-momentum  $P_{\text{Coul.,out}}^\mu$  of the electron under consideration. We see that it is not necessary to introduce some advanced field into the theory.

2) After having rejected the use of advanced quantities, let us now study, whether it is meaningful, to build up an equation of motion with the quantity  $P_{\text{tot}}^\mu$ , or  $P_{\text{tot}}^\mu - P_{\text{Coul.}}^\mu$ , either for finite  $s$  or in the asymptotic regions.

Usually<sup>5,19</sup>, one starts with the requirement that  $P_{\text{tot}}^\mu$  be a constant of the motion, first in comparing the asymptotic regions ( $P_{\text{tot,initial}}^\mu = P_{\text{tot,final}}^\mu$ ) but then in regarding every section of the world-line, too (the argumentation here is not falsificated by introducing some additional four-momentum of non-electromagnetic nature, in order to achieve mass renormalization). But we must not forget that the integration, resulting in  $P_{\text{tot}}^\mu$ , runs over the whole plane  $\sigma_\perp$  and takes into account therefore the sources of  $F_{\text{ext.}}^{\mu\nu}$ , as well in the interaction region (finite  $s$ ) as in the asymptotic region ( $s \rightarrow \pm\infty$ ). Clearly we must assume, that the sources of  $F_{\text{ext.}}^{\mu\nu}$  are electrons (or some other charged particles), and their contribution to the integral is of the same order of magnitude as the contribution of the electron under consideration! Thus the use of the quantity  $P_{\text{tot}}^\mu$  leads inevitably to a many-body-problem with electromagnetic interaction. Furthermore, the



quantity  $P_{\text{tot}}^\mu$  is not the total electromagnetic four-momentum of all charges in the universe, because the world-lines of these charges are, in general, not orthogonal to the plane  $\sigma_\perp$ , which defines  $P_{\text{tot}}^\mu$  due to (8).

Arrived at this point, we have to concede that an equation of motion cannot be derived without any new idea, exceeding Maxwell's theory. It is not difficult, however, to introduce a new assumption without any artificial constraint.

**Postulate:** The motion of the radiating electron in an external field takes such a course, that the total electromagnetic flux through the tube-surface  $(t)$  between two arbitrarily chosen hyperplanes  $\sigma_{\perp(1)}$  and  $\sigma_{\perp(2)}$  (see Fig. 5) vanishes, i. e.

$$\frac{1}{c} \int_1^2 T_{\text{tot}}^{\mu\nu} d^3 f_\nu = 0. \quad (38)$$

Clearly, this postulate is very plausible in the context of the present model, because we have regarded the interior of the tube  $(t)$  as inaccessible to any classical treatment and therefore we must not allow, that classical electromagnetic energy can be accumulated at any place in the interior of the tube.

In differential form, Eq. (38) reads

$$\frac{1}{c} \int_{\hat{\Omega}} T_{\text{tot}}^{\mu\nu} \frac{d^3 f_\nu}{ds} = 0. \quad (38 \text{ a})$$

We can bring this equation of motion in a form easier to survey by inserting (36), (5–6b) and the results (34–34b). We then find

$$\frac{d}{ds} \left\{ \frac{Z^2}{2 \varrho} \left[ \frac{4}{3} \hat{u}^\mu (\hat{u} u) - \frac{1}{3} u^\mu \right] \right\} = \mathcal{K}_{\text{el}}^\mu + \frac{2}{3} Z^2 (\hat{u} \dot{\hat{u}}) \hat{u}^\mu. \quad (39)$$

In (39) we have abbreviated

$$\mathcal{K}_{\text{el}}^\mu = + \int_{\hat{\Omega}} (T_{\text{int.}}^{\mu\nu} + T_{\text{ext.}}^{\mu\nu}) \frac{d^3 f_\nu}{ds}. \quad (39 \text{ a})$$

Though the expression on the right-hand side of (39 a) can be calculated, if the world-line and  $F_{\text{ext.}}^{\mu\nu}$  are given, in practical problems the knowledge of  $\mathcal{K}_{\text{el}}^\mu$  should be necessary only in some approximation. This question is dealt with in the next section.

Finally let us note that the equation of motion (39) has the plausible form

$$dP_b^\mu/d\tau = \mathcal{K}_{\text{el}}^\mu - dP_{\text{rad}}^\mu/d\tau.$$

## IX. Approximated forms of the equation of motion

Obviously, the unwieldiness of the exact equation of motion is conditioned by having taken account for the extension of the electron up to infinitely high order. The desired approximation is therefore expected to consist of some sort of expansion with respect to the extension-parameter  $(\Delta s)$ .

Of course, the first step in this direction is the expansion of  $\hat{u}^\mu$ ,  $\dot{\hat{u}}^\mu$  in the term  $dP_{\text{part}}^\mu/ds$  of Equation (39):

$$\hat{u}^\mu = u^\mu - \Delta s \cdot \dot{u}^\mu + \frac{1}{2} (\Delta s)^2 \ddot{u}^\mu \dots \quad (40)$$

and analogous for  $\dot{\hat{u}}$ . If we would insert the expansion in (39) and retain only terms of maximal order  $(\Delta s)^0$ , we would have found at once (apart from mass renormalization) the Lorentz-Dirac equation

$$\frac{Z^2}{2 \cdot \Delta s} \dot{u}^\mu = \mathcal{K}_{\text{el}}^\mu + \frac{2}{3} Z^2 [\ddot{u}^\mu + (\dot{u} \dot{u}) u^\mu].$$

In this order of approximation, one has of course

$$\mathcal{K}_{\text{el}}^\mu = Z F_{\text{ext.}}^{\lambda\mu} u_\lambda.$$

One would then have to conclude from this procedure, that the Lorentz-Dirac equation is the exact equation of motion for a point-like particle. A similar conclusion seems to be contained in Dirac's original work.

Let us proceed here somewhat more carefully by noting that the expansions like (40) are not complete, because the real trajectory of the extended particle depends itself on the extension. Therefore the quantities  $u^\lambda$ ,  $\dot{u}^\lambda$ ,  $\ddot{u}^\lambda$ , ... in (40) depend still on  $\Delta s$ . A second expansion becomes necessary:

$$\begin{aligned} z_{(s)}^\mu &= z_{(s)}^\mu + (\Delta s) \cdot z_{(1)}^\mu + (\Delta s)^2 \cdot z_{(2)}^\mu + \dots \\ u_{(s)}^\mu &= u_{(s)}^\mu + (\Delta s) \cdot u_{(1)}^\mu + (\Delta s)^2 \cdot u_{(2)}^\mu + \dots \end{aligned} \quad (41)$$

and as well

$$\mathcal{K}_{\text{el}}^\mu = \mathcal{K}_{(0)}^\mu + \Delta s \cdot \mathcal{K}_{(1)}^\mu + (\Delta s)^2 \mathcal{K}_{(2)}^\mu + \dots \quad (41 \text{ a})$$

Of course, we cannot require that the convergence radius of the expansion (41) extends over the whole trajectory, but in order to derive a local, approximated equation of motion it is only necessary that the expansion (41) holds in a differentially small interval  $I$  about some reference-point  $P$  on the tra-

jectory. As the small corrections  $z_{(s)}^\mu$  are not of interest but so is the sequence

$$\zeta_{(j)}^\mu = \sum_{k=0}^j z_{(s)}^\mu \cdot (\Delta s)^k$$

of approximated trajectories, steadily improved by the higher orders, we consider the tangent-vectors

$$w_{(j)}^\mu = \sum_{k=0}^j u_{(k)}^\mu \cdot (\Delta s)^k \quad (42)$$

to the approximated trajectories. If we are satisfied with the  $j$ -th approximation, we will consider  $w_{(j)}^\mu$  as unit-tangent vector to the  $j$ -th approximation of the trajectory. Obviously,  $w_{(j)}^\mu$  covers the influence of the extension up to the  $j$ -th order and  $(\Delta s)^l \cdot w_{(j)}^\mu$  is then of  $(l+k)$ -th order or  $(\dot{w} \dot{w})_{(j)}^{(k)}$ , e. g., is of  $(2j)$ -th order.

The expansions (40) and (41) are now inserted in (39) and the various orders of approximation  $j=1, 2, 3, \dots$  are studied successively, by substituting quantities like

$$(\Delta s)^n \cdot \frac{d^l u^\mu}{ds^l} \rightarrow (\Delta s)^n \cdot \frac{d^l w^\mu}{ds^l}$$

if one stops at the  $j$ -th step.

Let us look at the four lowest-order approximations:

$$j = -1.$$

Since  $\mathcal{K}_{\text{el}}^\mu$  has no contribution of order  $(\Delta s)^{-1}$ , we find

$$\frac{1}{c} \frac{Z^2}{2 \cdot \Delta s} \dot{w}_{(0)} = 0 \quad \text{or} \quad m_{\text{el}} \cdot \frac{dw_{(0)}}{d\tau} = 0 \quad (43)$$

with  $m_{\text{el}} c^2 = Z^2/2 \Delta s$ . We see that the particle is not influenced at all by the force field and therefore the lowest-order trajectory is a straight line, being the tangent to the real trajectory in the reference-point P:

$$w_{(0)}^\mu \equiv \left. \frac{dz^\mu}{ds} \right|_{\text{P}} \quad (44)$$

Clearly, all higher-order approximations must retain this correct tangent:

$$w_{(j)}^\mu = \left. \frac{dz^\mu}{ds} \right|_{\text{P}} \quad (45)$$

$$j = 0 \quad (Z^2/2 \Delta s) \dot{w}_{(1)}^\mu = \mathcal{K}_{(0)}^\mu. \quad (46)$$

$\mathcal{K}_{\text{el}}^\mu$  contributes in this order the well-known Lorentz-force

$$\mathcal{K}_{(0)}^\mu = Z F_{\text{ext}}^{\lambda\mu} w_{(0)\lambda}$$

and therefore we have in the interval I

$$m_{\text{el}} \cdot c \frac{dw_{(1)}^\mu}{d\tau} = Z F_{\text{ext}}^{\lambda\mu} w_{(0)\lambda} \quad (47)$$

and in the reference-point P [note (44)]:

$$m_{\text{el}} \cdot c \left. \frac{dw_{(1)}^\mu}{d\tau} \right|_{\text{P}} = Z F_{\text{ext}}^{\lambda\mu} \frac{dz_{\lambda}}{ds} \Big|_{\text{P}} \quad (48)$$

Equation (48) gives a first-order expression of the change of the tangent in P and in this sense the equation

$$m_{\text{el}} c dw_{(1)}^\mu/d\tau = Z F_{\text{ext}}^{\lambda\mu} w_{(0)\lambda} \quad (49)$$

describes the differential-geometrical behaviour of the true trajectory approximately in some point P. (49) is therefore a first approximation to the exact equation of motion (39), thus recovering Newton's second law in relativistic form.

$$j = 1$$

$$m_{\text{el}} \cdot c \frac{dw_{(2)}^\mu}{d\tau} - \frac{2}{3} \frac{Z^2}{c^2} \frac{d^2 w_{(1)}^\mu}{d\tau^2} = \mathcal{K}_{(1)}^\mu. \quad (50)$$

In the Appendix B we show that in  $\mathcal{K}_{(1)}^\mu$  no additional term to the Lorentz-force

$$\mathcal{K}_{(1)}^\mu = Z F_{\text{ext}}^{\lambda\mu} w_{(1)\lambda} \quad (51)$$

emerges [cf. (B 5)], if we neglect the variability of  $F_{\text{ext}}^{\lambda\mu}$  over the extension of the electron.

Regarding Eq. (50), one could be tempted to replace

$$\frac{d^2 w_{(1)}^\mu}{d\tau^2} \rightarrow \frac{d^2 w_{(2)}^\mu}{d\tau^2} \quad (52)$$

in order to obtain a differential equation for  $w_{(2)}^\mu$  ( $\rightarrow w_{(2)}^\mu$ ):

$$m_{\text{el}} \cdot c \frac{dw_{(2)}^\mu}{d\tau} - \frac{2}{3} \frac{Z^2}{c^2} \frac{d^2 w_{(2)}^\mu}{d\tau^2} = Z F_{\text{ext}}^{\lambda\mu} w_{(2)\lambda} \quad (53)$$

but this is a meaningless equation (multiply with  $w_{(2)\mu}$ ). Nevertheless the incorrect step (52), which destroys the homogeneity in the approximation order

in Eq. (50), shows explicitly how the unphysical effect of preacceleration penetrates into the theory: Eq. (53) reads in integro-differential form

$$m_{\text{el}} c \frac{dw^\mu}{d\tau} = \int_{\tau}^{\infty} \mathcal{K}_{(1)}^\mu \exp \left\{ -\frac{\tau' - \tau}{\Delta\tau} \right\} \frac{d\tau'}{\Delta\tau} \\ \left( \overline{\Delta\tau} = \frac{3}{4} \Delta\tau = \frac{3}{4} \frac{\Delta s}{c} \right).$$

From here we see very clearly that the unphysical effect of preacceleration is the punishment for having allowed the incorrect procedure (52). This will become more clear, when we shall try to derive the Lorentz-Dirac equation in the next step ( $j=2$ ).

Let us now derive the correct second-order approximation by substituting for  $d^2w^\mu/d\tau^2$  in (50) the value following from (47)

$$m_{\text{el}} c \frac{dw^\mu}{d\tau} - \frac{2}{3} \frac{Z^2}{c^2} \frac{1}{m_{\text{el}} c} \cdot Z \frac{dF_{\text{ext.}}^{\lambda\mu}}{d\tau} w_\lambda = Z F_{\text{ext.}}^{\lambda\mu} w_\lambda. \quad (51)$$

This equation is valid in the interval  $I$  and in the reference-point  $P$  we have ( $w^\mu \rightarrow w^\mu$ ):

$$m_{\text{el}} c \frac{dw^\mu}{d\tau} - \frac{4}{3} \Delta\tau Z \frac{dF_{\text{ext.}}^{\lambda\mu}}{d\tau} w_\lambda = Z F_{\text{ext.}}^{\lambda\mu} w_\lambda. \quad (54)$$

This equation represents the first improvement with respect to Newton's second law (49), but it is not a new equation: Eliezer<sup>10</sup> had postulated Eq. (54) as relativistic generalization of

$$m_{\text{el}} \frac{d\mathbf{v}}{dt} = Z \left( 1 + \frac{4}{3} \Delta\tau \frac{d}{dt} \right) \mathbf{E} \quad (54 \text{ a})$$

where  $\mathbf{E}$  is the usual electric field-strength. But he had rejected (54) at once, because this equation exhibits no radiation-damping if the electron is moving in constant, homogeneous fields. However, this fact does not trouble us here, because we know that (54) is not the exact equation but only the next higher approximation following Newton's second law.

**j = 2**

$$m_{\text{el}} c \frac{dw^\mu}{d\tau} - \frac{2}{3} \frac{Z^2}{c^2} \frac{d^2w^\mu}{d\tau^2} + \frac{1}{3} \frac{Z^2}{c^2} \Delta\tau \frac{d^3w^\mu}{d\tau^3} \\ = \mathcal{K}_{(2)}^\mu + \frac{2}{3} \frac{Z^2}{c^2} \left( \frac{dw^\lambda}{d\tau} \frac{dw_\lambda}{d\tau} \right) w^\mu. \quad (55)$$

For  $\mathcal{K}_{(2)}^\mu$  we can substitute here

$$\mathcal{K}_{(2)}^\mu = Z F_{\text{ext.}}^{\lambda\mu} w_\lambda^{(2)} \quad (56)$$

because the term with  $T_{\text{ext.}}^{\mu\nu}$  in (39 a) does not contribute anything in the desired order and the term with  $T_{\text{int.}}^{\mu\nu}$  must contain higher derivatives of  $w^\mu$ , which however vanish on account of (43). Now, from (55) we can obtain the Lorentz-Dirac-Eq. (1) if we proceed as follows:

- 1) neglect the term with  $d^3w^\mu/d\tau^3$ ,
- 2) replace  $d^2w^\mu/d\tau^2 \rightarrow d^2w^\mu/d\tau^2$ ,
- 3) replace  $dw^\lambda/d\tau \rightarrow dw^\lambda/d\tau$ .

But this procedure would indeed mean, that we collect terms of various orders [in  $(\Delta s)$ ] in a very inconsistent manner, and the resulting Eq. (1) lacks of any reliability! Of course, as under  $j=1$ , the existence of runaway solutions and preacceleration is due to this inconsistent procedure.

The correct third-order equation of motion is now obtained by inserting into Eq. (55) the values for  $d^2w^\mu/d\tau^2$ ,  $d^3w^\mu/d\tau^3$ ,  $dw^\lambda/d\tau$  iteratively from (43), (46), (50) and then considering the reference-point  $P(w^\mu \rightarrow w^\mu)$ :

$$m_{\text{el}} c \frac{dw^\mu}{d\tau} = Z \left\{ F_{\text{ext.}}^{\lambda\mu} + \frac{4}{3} (\Delta\tau) \frac{dF_{\text{ext.}}^{\lambda\mu}}{d\tau} + \frac{10}{9} (\Delta\tau)^2 \frac{d^2F_{\text{ext.}}^{\lambda\mu}}{d\tau^2} \right\} w_\lambda \quad (57)$$

$$+ \frac{8}{3} (\Delta s)^2 \{ -F_{\text{ext.}}^{\mu\lambda} F_{\text{ext.}}^{\nu\lambda} \lambda w_\nu + (F_{\text{ext.}}^{\alpha\beta} F_{\text{ext.}}^{\sigma\beta} w_\alpha w_\sigma) w^\mu \}.$$

This equation of motion covers correctly the finite-size effect up to order  $(\Delta s)^2$  and should replace the Lorentz-Dirac equation if practical problems have to be solved. Clearly, in (57) no preacceleration and no runaway solutions exist. We see that the extension parameter  $(\Delta s)$  plays the part of an interaction-parameter besides the charge  $Z$ . In this context it is interesting to note that due to (43) a point-like particle exhibits no interaction with an external field.

Finally, one peculiar feature of the new equation of motion has to be noted:

If we regard the second term on the right-hand side of (57):

$$\begin{aligned}
& -F^{ku} F^v_{\lambda} + (F^{ov} F^a_v w_o w_o) w^u \\
& = \{ \mathbf{E}^2 \cdot \mathbf{w}^0 - (\mathbf{E} \times \mathbf{B}) \cdot \mathbf{w} ; (\mathbf{E} \times \mathbf{B}) \mathbf{w}^0 + \mathbf{E}(\mathbf{E} \cdot \mathbf{w}) \\
& \quad + \mathbf{B}(\mathbf{B} \cdot \mathbf{w}) - \mathbf{B}^2 \mathbf{w} \} \\
& + [ -\mathbf{E}^2 w^{02} + 2(\mathbf{E} \times \mathbf{B}) \cdot \mathbf{w} w^0 + (\mathbf{E} \cdot \mathbf{w})^2 \\
& + (\mathbf{B} \cdot \mathbf{w})^2 - \mathbf{B}^2 w^2 ] \{ w^0 ; \mathbf{w} \} \quad (58)
\end{aligned}$$

we realize, that a radiation-damping term is now present if the motion takes place in a constant, homogeneous  $\mathbf{B}_{\perp}$ -field, the expression (58) then reducing to (note  $\mathbf{B} \perp \mathbf{w}$ ):

$$\{ -\mathbf{B}^2 \mathbf{w}^2 w^0 ; -\mathbf{B}^2 w^{02} \mathbf{w} \}.$$

This is the difference between Eqs. (54) and (57). But if the motion takes place along a constant, homogeneous  $\mathbf{E}$ -field, then the expression (58) reduces to zero and no radiation-reaction is present. This phenomenon is also observed in the theory of the Lorentz-Dirac equation (1) and in the theory of Mo + Papas<sup>13</sup>, as Huschilt<sup>20</sup> has noted recently. We consider this phenomenon to be a serious objection to both theories and the results of Rohrlich<sup>21</sup> seem to be very questionable in this respect!

However, in the framework of the theory presented here, the lack of radiation-damping in a constant, homogeneous  $\mathbf{E}$ -field cannot be an objection, because Eq. (57) is an approximative one, and we have good reason to believe that in the next higher approximation ( $j=3$ ) a radiation-reaction term arises, as well as such a term arose on the step (54) → (57) in the case of a homogeneous, constant  $\mathbf{B}$ -field.

## X. Summary of Results

1) We have constructed a covariant model of an electron of finite size, using Maxwell's theory of classical electromagnetism. The classical field equations are assumed to fail within the suitably defined "interior" of the electron, but no use of advanced quantities is made, in contrast to Dirac's approach and the current view in literature.

2) The geometrical implications of the new definition of the extension are studied, and it is shown that the mathematical problems involved can be overcome without taking the limit of vanishing size.

3) The model presented allows for a most natural decomposition of the total four-momentum of the electron into a bound part and an emitted part, each of them being exactly calculable. An explanation of the origin of the Schott-term can be given within the frame-work of the model.

(4) A covariant equation of motion to the extended electron, including radiation-damping, can be found, whereby Maxwell's theory has to be exceeded only by same simple and plausible postulate.

5) The exact equation of motion can be approximated iteratively without need of mass-renormalization, but the resulting equations of motion do not lead to the Lorentz-Dirac equation. The latter can be obtained only by a certain incorrect procedure.

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## Appendix A: The hypersurface-element $d^3f^i$ on the tube ( $t$ )

We imagine the tube ( $t$ ) to be composed of "stream-lines" having their rise from the parallel transport of the vector  $\hat{n}$  along the world-line of the source:

$$\delta \hat{n} / \delta s = 0. \quad (A1)$$

The tangent-vector  $\delta \xi$  to such a stream-line is obtainable from (12) by varying the vertex-point of  $l_{(\hat{z})}$  on the world-line [note (A1)]:

$$\delta \xi = \delta \hat{z} + \frac{\delta R_{\min}}{\delta s} \hat{n} \delta s = [\hat{u} + \dot{R}_{\min} \hat{n}] \delta s. \quad (A2)$$

We now project the tangent  $\delta \xi$  on to the unit-normal  $m$  of the intersection  $l_{(\hat{z})} \cdot \wedge \perp \sigma_{(z)}$  in the plane  $\sigma_{\perp(z)}$ :

$$(m \delta \xi) = (\hat{n} \delta \xi) / (\hat{n} u) - (u \delta \xi). \quad (A3)$$

The two scalar-products on the right-hand side of (A3) are obtainable by variation of the two equations (11). From (11 a) we get

$$\begin{aligned}
\delta(\xi - \hat{z})^2 &= 2(\xi - \hat{z}) \cdot (\delta \xi - \hat{u} \delta s) \\
&= 2 R_{\min} \hat{n} \cdot (\delta \xi - \hat{u} \delta s) = 0.
\end{aligned}$$

Therefore

$$(\hat{n} \cdot \delta \xi) = \delta s. \quad (A4)$$

And from (11 b) we get

$$\delta u \cdot (\xi - z) = \dot{u} \cdot (\xi - z) \delta s + u \cdot (\delta \xi - u \delta s) = 0.$$

Therefore

$$(u \cdot \delta \xi) = [1 - \dot{u} \cdot (\xi - z)] \delta s. \quad (A5)$$

(A 3) becomes with (A 4) and (A 5):

$$(m \delta \xi) = \frac{\delta s}{(\hat{n} u)} - [1 - \hat{u} \cdot (\xi - z)] \delta s. \quad (\text{A } 6)$$

From here we see that the projection  $(m \delta \xi)$  does not vanish in general, and therefore the streamlines are not orthogonal to the plane  $\sigma_{\perp}$ .

The calculation of the invariant hypersurface-element  $d^3f$  (writing  $d^3f^{\hat{\mu}} = f^{\hat{\mu}} \cdot d^3f$ ;  $f^{\hat{\mu}} f_{\hat{\mu}} = -1$ ) requires the projection  $\delta_{\perp} \xi$  of  $\delta \xi$  on to the two-dimensional “normal-plane” ( $u \vee m$ ) of the intersection  $l_{(z)} \wedge \sigma_{\perp(z)}$ , spanned by the two vectors  $u$  and  $m$ :

$$\delta_{\perp} \xi = (u \cdot \delta \xi) u - (m \cdot \delta \xi) m. \quad (\text{A } 7)$$

The “surface-area”  $d^3f$  of the “parallelogram”, which consists of the sphere-element  $d^2S = R_{\min}^2 d\hat{\Omega}$  as one face and  $\delta \xi$  as the other side, is

$$d^3f = d^2S \cdot |\delta_{\perp} \xi| \quad (\text{A } 8)$$

( $|\delta_{\perp} \xi|$  in Minkowskian measure).

After having found the invariant element  $d^3f$  in (A 8), we have now to look for the unit-normal  $f^{\hat{\mu}}$  to the tube ( $t$ ). As is well known from elementary differential geometry, the normal vector  $f^{\hat{\mu}}$  must lie in the two-dimensional plane ( $m \wedge u$ ), spanned by the “principal normal”  $m$  and the “binormal”  $u$ :

$$f \sim -(m \delta \xi) u + (u \delta \xi) m. \quad (\text{A } 9)$$

The coefficients of  $u$  and  $m$  have been chosen so that

$$(f \cdot \delta_{\perp} \xi) = 0$$

in order that  $f^{\hat{\mu}}$  be the normal to the tube ( $t$ ). After normalization to unity we obtain finally

$$f^{\hat{\mu}} = \frac{-(m \delta \xi) u + (u \delta \xi) m}{|\delta_{\perp} \xi|} \quad (\text{A } 10)$$

with

$$f^{\hat{\mu}} f_{\hat{\mu}} = -1$$

(note that  $|\delta_{\perp} \xi|$  is always positive).

(A 6), (A 5), (A 8) and (A 10) deliver the desired result

$$d^3f^{\hat{\nu}} = f^{\hat{\nu}} \cdot d^3f = ds [R_{\min}^2 / (\hat{n} u)] d\hat{\Omega} \{ -u^{\hat{\nu}} + [1 - \hat{u} \cdot (\xi - z)] \hat{n}^{\hat{\nu}} \}. \quad (\text{A } 11)$$

Of course, we can insert here  $R_{\min}$  from (13) and  $(\xi - z)$  from

$$\xi - z = -\mathcal{Z} + R_{\min} \hat{n}.$$

## Appendix B: The electromagnetic force $\mathcal{K}_{\text{el}}^{\mu}$ up to the first order

Since the hypersurface-element  $d^3f^{\hat{\mu}}$  is of at least second order [ $\sim O(\Delta s^2)$ ], a contribution to  $\mathcal{K}_{\text{el}}^{\mu}$  can come only from the interaction term in (39 a). One finds

$$-4\pi T_{\text{int}}^{\mu\nu} = 2 F_{\text{part}}^{(\mu\lambda)} F_{\text{ext}}^{\nu)} - \frac{1}{2} g^{\mu\nu} F_{\text{part}}^{\alpha\beta} F_{\alpha\beta, \text{ext}}. \quad (\text{B } 1)$$

The calculation of  $\int_{\hat{\Omega}} T_{\text{int}}^{\mu\nu} d^3f^{\hat{\nu}}/ds$  in (39 a) requires therefore the knowledge of

$$\int_{\hat{\Omega}} F_{\text{part}}^{\mu\lambda} d^3f^{\hat{\nu}}/ds \quad (\text{B } 2)$$

with  $F_{\text{part}}^{\mu\lambda}$  taken from (3). Expanding the integrand about the point  $\hat{z}$  up to order  $(\Delta s)^1$ , one finds

$$F_{\text{part}}^{\mu\lambda} d^3f^{\hat{\nu}}/ds \cong 2 Z d\hat{\Omega} \{ \hat{u}^{[\mu} \hat{v}^{\lambda]} \hat{v}^{\nu]} + \Delta s \cdot \hat{u}^{[\mu} \hat{v}^{\lambda]} \hat{v}^{\nu]} - \Delta s (\hat{v} \hat{u}) \hat{u}^{[\mu} \hat{v}^{\lambda]} \hat{u}^{\nu]} + O_{(\Delta s^2)} \}.$$

If we now perform the integration (B 2), we observe

$$\int_{\hat{\Omega}} \frac{d\hat{\Omega}}{4\pi} \hat{v}^{\lambda} \hat{v}^{\nu} = \frac{1}{3} [\hat{u}^{\lambda} \hat{u}^{\nu} - g^{\lambda\nu}],$$

$$\int_{\hat{\Omega}} \frac{d\hat{\Omega}}{4\pi} (\hat{v} \hat{u}) \hat{v}^{\lambda} = -\frac{1}{3} \hat{u}^{\lambda}$$

and then obtain

$$\frac{1}{4\pi} \int_{\hat{\Omega}} F_{\text{part}}^{\mu\lambda} \frac{d^3f^{\hat{\nu}}}{ds} \cong \frac{2}{3} Z g^{\nu[\mu} (\hat{u}^{\lambda]} + \hat{u}^{\lambda]} \cdot \Delta s) + O_{(\Delta s^2)} = \frac{2}{3} Z g^{\nu[\mu} u^{\lambda]} + O_{(\Delta s^2)}. \quad (\text{B } 4)$$

With (B 1) and (B 4) one gets

$$-\int T_{\text{int}}^{\mu\nu} \cdot d^3f^{\hat{\nu}}/ds = Z F_{\text{ext}}^{\mu\lambda} u_{\lambda} + O_{(\Delta s^2)} \quad (\text{B } 5)$$

which was set out to proof. Note, that we have not taken into account the variability of  $F_{\text{ext}}^{\mu\lambda}$  over the range of extension of the electron!

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